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# Bound states due to a strong $\delta$ interaction supported by a curved surface

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**Abstract**

We study the Schrödinger operator $-\Delta - \alpha\delta(x - \Gamma)$ in $L^2(\mathbb{R}^3)$ with a $\delta$ interaction supported by an infinite non-planar surface $\Gamma$ which is smooth and admits a global normal parametrization with a uniformly elliptic metric. We show that if $\Gamma$ is asymptotically planar in a suitable sense and $\alpha > 0$ is sufficiently large, this operator has a non-empty discrete spectrum and derive an asymptotic expansion of the eigenvalues in terms of a 'two-dimensional' comparison operator determined by the geometry of the surface $\Gamma$.

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## 1. Introduction

The fact that a quantum particle localized in a curved infinitely extended region can have bound states has been known for more than a decade (cf [EŠ, DE] and references therein). It was first demonstrated for curved hard-wall strips and tubes. The analogous problem in curved layers is more complicated and the existence of curvature-induced bound states has been demonstrated only recently [DEK]. In addition, the sufficient conditions known so far apply to particular classes of layers and lack therefore the universal character of the 'one-dimensional' existence result noticed first in [GJ].

Another recent observation concerns the fact that the effect can persist if the transverse Dirichlet condition confinement is replaced by a weaker one. This is important if we want to apply the conclusions to models of quantum wires and similar structures in which the confinement is realized by a finite potential step at an interface of two different semiconductor materials. As a consequence, an electron can pass between two parts of a quantum wire also by tunnelling through the classically forbidden region separating them.

To make the task more feasible, one can study the idealized situation in which the structure is infinitely thin and the Hamiltonian is formally written as $-\Delta - \alpha\delta(x - \Gamma)$, where $\alpha$ is a real parameter. It can be interpreted as a limiting case of a transverse confinement by a deep and narrow potential well, at least as long as the codimension of the manifold $\Gamma$ is 1. The existence of a nontrivial discrete spectrum has been proved in this setting if $\Gamma$ is a curve in $\mathbb{R}^2$ [EI] and $\mathbb{R}^3$ [EK] which is asymptotically straight but not a straight line and satisfies suitable regularity conditions; the analogous result also holds for curved arrays of point interactions [Ex1]. The argument is in all the listed cases based on an explicit expression of the resolvent: one can check that the curvature gives rise to perturbation of the straight-line Birman–Schwinger operator which is Hilbert–Schmidt and of a definite sign.

As in the case of a hard-wall confinement the problem becomes more complicated if the region to which the particle is localized is generated by a surface in $\mathbb{R}^3$. The above-mentioned method does not generalize directly to such a situation, because surfaces lack—in distinction to curves—a natural parametrization which would allow us to 'iron' them into a plane. This is why we address the question in this paper using a different method which will make it possible to establish the existence of curvature-induced bound states provided the attractive $\delta$ interaction in the Hamiltonian is strong enough. The idea is borrowed from [EY1, EY2] and is based on a combination of Dirichlet–Neumann bracketing with the use of suitable curvilinear coordinates in the vicinity of the surface $\Gamma$, which is supposed to be asymptotically planar with a uniformly elliptic metric. In this way we not only prove in theorem 4.3 the existence of a nontrivial discrete spectrum, but also obtain an asymptotic expansion of the eigenvalues as $\alpha \to \infty$ in terms of a suitable 'two-dimensional' comparison operator determined by the geometry of $\Gamma$ (cf (2.2)). In the appendix we give precise meaning to the above statement about the relation between our Hamiltonian and the operator with a deep and narrow potential well centred at the surface.

## 2. Main results

Let us start by summarizing our main results in more precise terms. As we have said the Hamiltonians we will study are the Schrödinger operators with the singular perturbations supported by an infinite surface $\Gamma$ in $\mathbb{R}^3$ corresponding to the formal expression

$$-\Delta - \alpha\delta(x - \Gamma) \tag{2.1}$$

where $\alpha > 0$ is independent of $x$. A general way to give meaning to (2.1) as a well-defined self-adjoint operator (denoted by $H_{\alpha,\Gamma}$) is to employ the sum of quadratic forms; we will do that in section 3.2.1.

The form sum definition works under rather weak assumptions about the regularity of $\Gamma$. For further purposes we have to restrict the class of surfaces: the main results of the paper will be derived for $\Gamma$ which is assumed to be

- $C^4$ smooth and admitting a global normal parametrization with a uniformly elliptic metric tensor—see (3.1) and (3.9),
- without 'near-intersections'—assumption (a$\Gamma$1) of section 3.1,
- asymptotically planar—assumption (a$\Gamma$2) or a stronger hypotesis (a$\Gamma 2'$) of section 4.

Our goal is to investigate spectral properties of $H_{\alpha,\Gamma}$ in the asymptotic regime when $\alpha$ is large. To this aim we employ the comparison operator

$$S = -\Delta_\Gamma - \tfrac{1}{4}(k_1 - k_2)^2 \tag{2.2}$$

where $\Delta_\Gamma$ is the Laplace–Beltrami operator on $\Gamma$ and $k_1, k_2$ are the principal curvatures of $\Gamma$. If both $k_1, k_2$ are identically zero, i.e. if $\Gamma$ is a plane, it is easy to show that operator $H_{\alpha,\Gamma}$ has purely absolutely continuous spectrum given by $\sigma_{ac}(H_{\alpha,\Gamma}) = \left[-\frac{1}{4}\alpha^2, \infty\right)$; it is sufficient to employ separation of variables and use the spectrum of one-dimensional $\delta$ interaction.

The aim of this paper is to prove that a 'local' deformation of $\Gamma$ leads to the existence of bound states for large enough $\alpha$. More precisely, we will show in theorem 4.3 that if $\Gamma$ is not a plane and satisfies the above assumptions, then

- the bottom of the essential spectrum does not lie below $\epsilon(\alpha)$, where $\epsilon(\cdot)$ is a function such that $\epsilon(\alpha) \to -\frac{1}{4}\alpha^2$ as $\alpha \to \infty$,
- there exists at least one isolated point of the spectrum below the threshold of the essential spectrum for all sufficiently large $\alpha$, and moreover, the eigenvalues of $H_{\alpha,\Gamma}$ have the following asymptotical expansion,
$$\lambda_j(\alpha) = -\tfrac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1}\log\alpha)$$
as $\alpha \to \infty$, where $\mu_j$ is the *j*th eigenvalue of $S$. The existence of a non-empty discrete spectrum alone can be proved even without the uniform ellipticity assumption (see remark 4.6).

Let us remark that the existence of a global normal parametrization is not necessary for the derivation of the above asymptotic formula (cf [Ex2]). It plays an important role, however, when we prove that the discrete spectrum is non-empty by comparison with a suitable two-dimensional Schrödinger operator. An analogy with curved Dirichlet layers [CEK] suggests that bound states may exist even for some classes of surfaces $\Gamma$ which are not simply connected, but the proof of this conjecture remains an open problem.

## 3. Preliminaries

### *3.1. Geometry of the surface and its neighbourhood*

Let $\Gamma \subset \mathbb{R}^3$ be an infinite $C^4$ smooth surface which admits a global normal parametrization (we refer to [Kl] for the geometric notions used below). This requires the existence of a point $o \in \Gamma$ such that the exponential map $\exp_o : T_o\Gamma \to \Gamma$ is a diffeomorphism. Given an orthonormal basis $\{e_1(o), e_2(o)\}$ in $T_o\Gamma$ we introduce the map $\gamma \equiv \gamma_o : T_o\Gamma \cong \mathbb{R}^2 \to \Gamma$ defined by

$$s \equiv (s_1, s_2) \to \exp_o\left(\sum_i s_i e_i(o)\right) \tag{3.1}$$

which determinates the said normal parametrization.

We denote by $g_{\mu\nu}$ the surface metric tensor in normal coordinates, i.e. $g_{\mu\nu} = \gamma_{,\mu} \cdot \gamma_{,\nu}$, and use the standard convention $g^{\mu\nu} = (g_{\mu\nu})^{-1}$. By means of the determinant $g := \det g_{\mu\nu}$ we define the invariant element of surface $\mathrm{d}\Gamma = g^{1/2}\,\mathrm{d}^2 s$. Furthermore, the tangent vectors $\gamma_{,\mu}$ are linearly independent and their cross product $\frac{\gamma_{,\mu}\times\gamma_{,\nu}}{|\gamma_{,\mu}\times\gamma_{,\nu}|}$ defines the unit normal field $n(s)$ on $\Gamma$.

The extrinsic properties of surface can be characterized in terms of the Weingarten tensor obtained by raising the index in the second fundamental form,

$$h_\mu{}^\nu := -n_{,\mu} \cdot \gamma_{,\sigma} g^{\sigma\nu}. \tag{3.2}$$

The eigenvalues of $h_\mu{}^\nu$ are the pricipal curvatures $k_1, k_2$; by means of them we define the Gauss curvature $K$ and mean curvature $M$:

$$K = \det h_\mu{}^\nu = k_1 k_2 \qquad M = \tfrac{1}{2}\mathrm{Tr}\, h_\mu{}^\nu = \tfrac{1}{2}(k_1 + k_2). \tag{3.3}$$

It follows immediately from the above formulae that

$$K - M^2 = -\tfrac{1}{4}(k_1 - k_2)^2. \tag{3.4}$$

Global quantities characterizing $\Gamma$ are the total Gauss curvature $\mathcal{K} := \int_\Gamma K\,\mathrm{d}\Gamma$ which exists provided $K \in L^1(\Gamma, \mathrm{d}\Gamma)$ (if this is not true the integral can sometimes exist as the principal value with respect to the geodesic radius—see remark 3.1b) and the total mean curvature $\mathcal{M} := \left(\int_\Gamma M^2\,\mathrm{d}\Gamma\right)^{1/2}$ (we set $\mathcal{M} = \infty$ if $M \notin L^2(\Gamma, \mathrm{d}\Gamma)$).

The approximation methods which we will use in further discussion force us to impose additional assumptions on $\Gamma$ which will allow us to work in a neighbourhood of the surface. Given $\delta > 0$ we consider the layer $\Omega_\delta$ built over $\Gamma$ and defined by virtue of the map

$$\mathcal{L} : \mathrm{D}_\delta \ni q \equiv (s, u) \to \gamma(s) + un(s) \qquad \mathrm{D}_\delta := \{(s, u) : s \in \mathbb{R}^2, u \in (-\delta, \delta)\}. \tag{3.5}$$

The fact that $\gamma$ is a diffeomorphism excludes automatically the possibility of self-intersections of $\Gamma$. Henceforth we also assume that $\Gamma$ does not admit 'near-intersections'; this is guaranteed by the following requirement:

(a$\Gamma$1) there exists $d > 0$ such that the map $\mathcal{L} : \mathrm{D}_d \to \Omega_d$ is injective.

Let us stress that this is a restriction imposed on the *global* geometry of $\Gamma$ which does not follow, e.g., from the mere decay of the curvatures expressed by the assumption (a$\Gamma$2) of section 4 (although it is implied by (a$\Gamma 2'$)). An example is easily constructed using deformations of the plane in the form of a smooth 'bubble' with a narrow 'bottleneck'; it is sufficient to consider a suitable array of such deformations with properly changing parameters.

Using the parametrization (3.5) we can find the metric tensor of $\Omega_d$ regarded as a submanifold in $\mathbb{R}^3$,

$$G_{ij} = \begin{pmatrix} (G_{\mu\nu}) & 0 \\ 0 & 1 \end{pmatrix} \qquad G_{\mu\nu} = \left(\delta_\mu^\sigma - uh_\mu{}^\sigma\right)\left(\delta_\sigma^\rho - uh_\sigma{}^\rho\right)g_{\rho\nu}.$$

In particular, the volume element of $\Omega_d$ is given by $\mathrm{d}\Omega := G^{1/2}\,\mathrm{d}^2s\,\mathrm{d}u$, where $G := \det G_{ij}$ takes the following form:

$$G = g[(1 - uk_1)(1 - uk_2)]^2 = g(1 - 2Mu + Ku^2)^2. \tag{3.6}$$

For the sake of brevity we employ the shorthand $\xi(s, u) \equiv 1 - 2M(s)u + K(s)u^2$. Moreover, we will use Greek notation for the range $(1, 2)$ of indices and Latin for $(1, 2, 3)$. The index numbering $(1, 2, 3)$ here refers to the coordinates $(s_1, s_2, u)$.

With a future purpose in mind we state some useful estimates. Suppose that $k_1, k_2$ are uniformly bounded (in fact we will assume more—see (a$\Gamma$2) below) and put

$$\varrho := (\{\max \|k_1\|_\infty, \|k_2\|_\infty\})^{-1}.$$

Is easy to verify that for $d < \varrho$ the following inequalities are satisfied in the layer neighbourhood $\Omega_d$,

$$C_-(d) \leqslant \xi \leqslant C_+(d) \tag{3.7}$$

where $C_\pm(d) := (1 \pm d\varrho^{-1})^2$. Consequently, we have

$$C_-(d)g_{\mu\nu} \leqslant G_{\mu\nu} \leqslant C_+(d)g_{\mu\nu}. \tag{3.8}$$

To make use of the last inequality we need information about the metric of the surface. To prove our main result we will require the uniform ellipticity of the metric tensor $g_{\mu\nu}$, i.e. we suppose that there exist positive constants $c^+, c^-$ such that

$$c^-\delta_{\mu\nu} \leqslant g_{\mu\nu} \leqslant c^+\delta_{\mu\nu} \tag{3.9}$$

is satisfied as a matrix inequality.

**Remarks 3.1.**

(a) Combining (3.7) and (3.6) one can check that the uniform boundedness of $k_1$, $k_2$ (together with the injectivity given by (a$\Gamma$1)) ensures that the map $\mathcal{L} : \mathrm{D}_d \to \Omega_d$ is diffeomorphic if $d < \varrho$.

(b) By means of the change of variables $\phi$ given by

$$s_1(r, \upsilon) = r\cos\upsilon \qquad s_2(r, \upsilon) = r\sin\upsilon$$

we pass to the geodesic polar coordinates (g.p.c.) $(r, \upsilon) = (y_1, y_2)$. In this notation $r = r(s) = \left(s_1^2 + s_2^2\right)^{1/2}$ determines the geodesic radius. The metric tensor

$$\tilde{g}_{\mu\nu}(y) = \sum_{\sigma\rho} \frac{\partial s_\sigma}{\partial y_\mu}\frac{\partial s_\rho}{\partial y_\nu} g_{\sigma\rho}(\phi(y))$$

acquires in the g.p.c. the diagonal form $\tilde{g}_{\mu\nu} = \mathrm{diag}(1, \rho^2)$, where $\rho$ satisfies the Jacobi equation

$$\ddot{\rho}(r, \upsilon) + K(r, \upsilon)\rho(r, \upsilon) = 0 \qquad \rho(0, \upsilon) = 0 \qquad \dot{\rho}(0, \upsilon) = 1. \tag{3.10}$$

(c) There exist various sufficient conditions for (3.9). For instance, if $\Gamma$ is a radially symmetric surface and $K \in L^1(\Gamma, \mathrm{d}\Gamma)$ but $\mathcal{K} \neq 2\pi$ then it is easy to show from (3.10) that there exist positive constants $\tilde{c}^+$, $\tilde{c}^-$ such that

$$\tilde{c}^- \tilde{g}^0_{\mu\nu} \leqslant \tilde{g}_{\mu\nu} \leqslant \tilde{c}^+ \tilde{g}^0_{\mu\nu} \tag{3.11}$$

where $\tilde{g}^0_{\mu\nu}$ is the metric tensor of a plane in polar coordinates; in other words, $\tilde{g}^0_{\mu\nu} = \mathrm{diag}(1, r^2)$, which in turn implies (3.9). This class of surfaces includes, for instance, any hyperboloid of revolution. For surfaces without radial symmetry a different sufficient condition for (3.9) is needed, e.g., $\|K\|_{L^1(\Gamma,\mathrm{d}\Gamma)} < 2\pi$. For another sufficient condition see remark 4.2b.

*3.2. Schrödinger operators with singular perturbation supported by the surface* $\Gamma$

*3.2.1. Construction of the Hamiltonian.* The Hamiltonians we will be interested in are Schrödinger operators with perturbations supported by $\Gamma$. A general way to construct such operators is to employ the form sum technique. Let us define the measure $\mu$ by

$$\mu \equiv \mu_\Gamma : \mu_\Gamma(B) := \mathrm{vol}(\mathrm{B} \cap \Gamma)$$

for each Borel set $B$ in $\mathbb{R}^3$, where $\mathrm{vol}(\cdot)$ is a two-dimensional Hausdorff measure on $\Gamma$. Using the fact that the map $\gamma$ is a diffeomorphism and making use of theorem 4.1 in [BEKŠ] it is easy to check that $\mu$ belongs to the generalized Kato class. Consequently, the embedding operator

$$I_\mu \psi = \psi \qquad I_\mu : \mathcal{S}(\mathbb{R}^3) \subset W^{2,1}(\mathbb{R}^3) \to L^2(\mathbb{R}^3, \mu) \equiv L^2(\mu)$$

is continuous and it can be extended to the whole space $W^{2,1}(\mathbb{R}^3)$. Clearly, we have the natural identification $L^2(\mu) \cong L^2(\mathbb{R}^2, \mathrm{d}\Gamma)$.

With this prerequisite we are able to construct the mentioned class of operators. Given $\alpha > 0$ we define the quadratic form

$$\eta_\alpha[\psi] \equiv \eta_\alpha(\psi, \psi) = (\nabla\psi, \nabla\psi) - \alpha \int_{\mathbb{R}^3} |I_\mu \psi(x)|^2 \, \mathrm{d}\mu(x) \qquad \psi \in W^{2,1}(\mathbb{R}^3)$$

where $(\cdot, \cdot)$ stands for the scalar product in $L^2(\mathbb{R}^3)$. It follows from theorem 4.2 of [BEKŠ] that $\eta_\alpha$ is bounded below and closed; therefore the associated operator $H_{\alpha,\Gamma} \equiv H_\alpha$ is bounded below and self-adjoint in $L^2(\mathbb{R}^3)$ and can be interpreted as the self-adjoint realization of the expression (2.1).

There is an alternative definition of $H_\alpha$ in terms of the boundary conditions on $\Gamma$; it is more illustrative because it shows that in the direction transverse to the surface the interaction is nothing other than a $\delta$ potential. For the surface $\Gamma$ with the properties specified in the previous section we consider the Laplace operator

$$(\dot{H}_\alpha\psi)(x) = -\Delta\psi(x) \qquad x \in \mathbb{R}^3\backslash\Gamma$$

with the domain consisting of functions from $C_0(\mathbb{R}^3) \cap W^{2,2}(\mathbb{R}^3\backslash\Gamma)$ and having a jump of the normal derivative on $\Gamma$ given by

$$\frac{\partial\psi}{\partial n}(x)\bigg|_+ - \frac{\partial\psi}{\partial n}(x)\bigg|_- = -\alpha\psi(x).$$

It is easy to verify that $\dot{H}_\alpha$ is e.s.a. and that by the Green formula it reproduces the form $\eta_\alpha$; therefore the closure of $\dot{H}_\alpha$ coincides with $H_\alpha$.

*3.2.2. Approximation by scaled potentials.* Before proceeding further, let us say a few words about the interpretation of the operator $H_\alpha$. If $\Gamma$ is smooth we can employ the standard approximation of the $\delta$ interaction by a family of squeezed potentials. To show this let us consider again the layer neighbourhood $\Omega_d$ of $\Gamma$ where $d < \varrho$. Given $W \in L^\infty(-1, 1)$ we define the scaled potentials with the support on $\Omega_d$, i.e.

$$V_d(x) := \begin{cases} 0 & \text{if} \quad x \notin \Omega_d \\ -\frac{1}{d}W\left(\frac{u}{d}\right) & \text{if} \quad x \in \Omega_d \end{cases}$$

and associate with them the operators

$$H_d(W) := -\Delta + V_d : D(\Delta) \to L^2(\mathbb{R}^3)$$

where $-\Delta : D(\Delta) \to L^2(\mathbb{R}^3)$ is the Laplace operator. Since the potentials are bounded the operators $H_d(W)$ are also self-adjoint with the domain $D(\Delta)$. This family approaches $H_\alpha$ as $d \to 0$ in the following sense:

**Theorem 3.2.** *Let the surface* $\Gamma$ *satisfy* (a$\Gamma$1). *Then* $H_d(W) \to H_\alpha$ *as* $d \to 0$, *where* $\alpha = \int_{-1}^{1} W(t)\,\mathrm{d}t$, *in the norm-resolvent sense.*

The proof is postponed to the appendix.

*3.2.3. Schrödinger operators with the perturbation on* $\Gamma$ *in the vicinity of the surface.* Let us return to our main subject. Our strategy is to estimate (the negative spectrum of) the operator $H_\alpha$ using operators acting in the layer neighbourhood of $\Gamma$. For the set $\Omega_d$ we define the quadratic forms $\eta_\alpha^+[\psi]$, $\eta_\alpha^-[\psi]$ with the domains $D(\eta_\alpha^+) = W_0^{2,1}(\Omega_d)$, $D(\eta_\alpha^-, ) = W^{2,1}(\Omega_d)$ which act as

$$\|\nabla\psi(x)\|^2_{L^2(\Omega_d)} - \alpha\int_{\mathbb{R}^3} |I_\mu\psi(x)|^2\,\mathrm{d}\mu.$$

Since the forms $\eta_\alpha^\pm$ are closed the associated operators $H_\alpha^\pm$ are self-adjoint in $L^2(\Omega_d)$. Now the Dirichlet–Neumann bracketing [RS] trick yields the estimate

$$-\Delta^N_{\Sigma_d} \oplus H_\alpha^- \leqslant H_\alpha \leqslant -\Delta^D_{\Sigma_d} \oplus H_\alpha^+ \qquad \Sigma_d \equiv \mathbb{R}^3\backslash\overline{\Omega}_d \tag{3.12}$$

and as long we are interested in the negative point of the spectrum, we may take into account $H_\alpha^\pm$ only, because the 'exterior' operators $\Delta^D_{\Sigma_d}$, $\Delta^N_{\Sigma_d}$ are positive by definition.

It is useful to treat the estimating operators $H_\alpha^\pm$ in the coordinates $q = (s, u)$. We pass to the curvilinear coordinates by means of the unitary transformation

$$U\psi = \psi \circ \mathcal{L} : L^2(\Omega_d) \to L^2(\mathrm{D}_d, \mathrm{d}\Omega).$$

We denote by $(\cdot,\cdot)_G$ the scalar product in the space $L^2(\mathrm{D}_d, \mathrm{d}\Omega)$, then the operators $UH_\alpha^+U^{-1}, UH_\alpha^-U^{-1}$ living in $L^2(\mathrm{D}_d, \mathrm{d}\Omega)$ are associated with the forms $\psi \mapsto \eta_\alpha^+[U\psi]$, $\eta_\alpha^-[U\psi]$ having the value

$$(\partial_i\psi, G^{ij}\partial_j\psi)_G - \alpha \int_\Gamma |\psi(s,0)|^2 \,\mathrm{d}\Gamma \tag{3.13}$$

which differ by their domains, $W_0^{2,1}(\mathrm{D}_d, \mathrm{d}\Omega)$ and $W^{2,1}(\mathrm{D}_d, \mathrm{d}\Omega)$, for the $\pm$ sign, respectively. Since the functions belonging to these spaces are not necessarily continuous, the expression $\psi(s,0)$ in (3.13) can be given a meaning using the trace mapping from $W_0^{2,1}(\mathrm{D}_d, \mathrm{d}\Omega)$ or $W^{2,1}(\mathrm{D}_d, \mathrm{d}\Omega)$ to $L^2(\Gamma, \mathrm{d}\Gamma)$. For convenience we will use in the following the same notation for $H_\alpha^\pm$, $\eta_\alpha^\pm$, and its unitary 'shifts' to the space $L^2(\mathrm{D}_d, \mathrm{d}\Omega)$.

It is also useful to remove the factor $\xi$ from the weight $G^{1/2}$ in space $L^2(\mathrm{D}_d, \mathrm{d}\Omega)$. This can be done by means of another unitary transformation,

$$\hat{U}\psi = \xi^{1/2}\psi : L^2(\mathrm{D}_d, \mathrm{d}\Omega) \to L^2(\mathrm{D}_d, \mathrm{d}\Gamma\,\mathrm{d}u). \tag{3.14}$$

We will denote the scalar product in $L^2(\mathrm{D}_d, \mathrm{d}\Gamma\,\mathrm{d}u)$ by $(\cdot,\cdot)_g$.

The operators

$$B_\alpha^+ := \hat{U}H_\alpha^+\hat{U}^{-1} \qquad B_\alpha^- := \hat{U}H_\alpha^-\hat{U}^{-1}$$

acting in $L^2(\mathrm{D}_d, \mathrm{d}\Gamma\,\mathrm{d}u)$ are associated with the forms $b_\alpha^\pm$ given by $b_\alpha^+[\psi] := \eta_\alpha^+[\hat{U}^{-1}\psi]$ and $b_\alpha^-[\psi] := \eta_\alpha^-[\hat{U}^{-1}\psi]$. By a straightforward computation we get

$$b_\alpha^+[\psi] = \left(\partial_\mu\psi, G^{\mu\nu}\partial_\nu\psi\right)_g + ((V_1+V_2)\psi,\psi)_g + \|\partial_3\psi\|_g^2 - \alpha\int_\Gamma |\psi(s,0)|^2\,\mathrm{d}\Gamma$$

$$b_\alpha^-[\psi] = b_\alpha^+[\psi] + \int_\Gamma \zeta(s,d)|\psi(s,d)|^2\,\mathrm{d}\Gamma - \int_\Gamma \zeta(s,-d)|\psi(s,-d)|^2\,\mathrm{d}\Gamma$$

for $\psi$ from $W_0^{2,1}(\mathrm{D}_d, \mathrm{d}\Gamma\,\mathrm{d}u)$ and $W^{2,1}(\mathrm{D}_d, \mathrm{d}\Gamma\,\mathrm{d}u)$, respectively, where $\zeta := \frac{M-Ku}{\xi}$ and

$$V_1 := g^{-1/2}\left(g^{1/2}G^{\mu\nu}J_{,\nu}\right)_{,\mu} + J_{,\mu}G^{\mu\nu}J_{,\nu} \qquad V_2 := \frac{K-M^2}{\xi^2} \qquad J := \frac{1}{2}\ln\xi. \tag{3.15}$$

## 4. Curvature-induced bound states

Let us now turn to the spectral analysis of $H_\alpha$. The main tool we will use to establish the existence of isolated points of spectrum of $H_\alpha$ for large $\alpha$ is the Dirichlet–Neumann bracketing (3.12). The idea is to construct operators $H_\alpha^\pm$ in the neighbourhood depending on parameter $\alpha$, i.e. $d = d(\alpha)$ such that $d(\alpha) \to 0$ as $\alpha \to \infty$, which would provide us with a sufficiently exact spectral approximation for $H_\alpha$. As usual working with a minimax-type argument we have to localize first the bottom of the essential spectrum.

### *4.1. Essential spectrum*

Let us start with the case when $\Gamma$ is a plane in $\mathbb{R}^3$. Then the translational invariance allows us to separate the variables showing thus that

$$\sigma(H_\alpha) = \sigma_{ac}(H_\alpha) = \left[-\tfrac{1}{4}\alpha^2, \infty\right).$$

Next we assume that the surface $\Gamma$ admits a deformation which is localized in the following sense:

$$(\mathrm{a}\Gamma 2) \quad K, M \to 0 \text{ as the geodesic radius } r \to \infty$$

The main result of this part says that the bottom of $\sigma_{\mathrm{ess}}(H_\alpha)$ can be pushed down by the deformation at most by a quantity which vanishes as $\alpha \to \infty$.

**Theorem 4.1.** *Let $\alpha > 0$ and suppose that the surface $\Gamma$ satisfies* (a$\Gamma$1), (a$\Gamma$2). *Then*

$$\sigma_{\mathrm{ess}}(H_\alpha) \subseteq [\epsilon(\alpha), \infty) \tag{4.1}$$

*where $\epsilon(\cdot)$ is a function such that*

$$\epsilon(\alpha) \to -\frac{\alpha^2}{4} \qquad as \quad \alpha \to \infty.$$

To prove the above theorem we will need a statement which follows directly from lemma 4.5: there exist constants $C, C_N$ such that

$$\int_{-d}^{d} \left| \frac{\mathrm{d} f(u)}{\mathrm{d} u} \right|^2 \mathrm{d} u - \alpha |f(0)|^2 \geqslant \left( -\frac{\alpha^2}{4} - C_N \alpha^2 \,\mathrm{e}^{-\alpha d/2} \right) \|f\|^2_{L^2(-d,d)} \tag{4.2}$$

with $f \in W^{2,1}(-d, d)$, holds for $\alpha > Cd^{-1}$.

**Proof.** Let us first note that the inclusion (4.1) is equivalent to

$$\inf \sigma_{\mathrm{ess}}(H_\alpha) \geqslant \epsilon(\alpha). \tag{4.3}$$

In view of (3.12) and positivity of $-\triangle^N_{\Sigma_d}$, inequality (4.3) follows from

$$\inf \sigma_{\mathrm{ess}}(H^-_\alpha) \geqslant \epsilon(\alpha) \tag{4.4}$$

where $H^-_\alpha$ acts in $L^2(\Omega_d)$, $d < \varrho$. Now we can proceed in analogy with the proof of theorem 4.1 in [DEK]. Let us divide the surface $\Gamma$ into two components $\Gamma^{\mathrm{int}}_\tau := \{s \in \Gamma : r(s) < \tau\}$ and $\Gamma^{\mathrm{ext}}_\tau := \Gamma \backslash \overline{\Gamma}^{\mathrm{int}}_\tau$. This gives rise to the division of the layer neighbourhood into $\mathrm{D}^{\mathrm{int}}_\tau$ and $\mathrm{D}^{\mathrm{ext}}_\tau$, where $\mathrm{D}^{\mathrm{int}}_\tau := \{(s, u) \in \mathrm{D}_d : s \in \Gamma^{\mathrm{int}}_\tau\}$ and $\mathrm{D}^{\mathrm{ext}}_\tau := \mathrm{D}_d \backslash \overline{\mathrm{D}}^{\mathrm{int}}_\tau$. Let us consider the Neumann decoupled operators

$$H^{-,\mathrm{int}}_{\alpha,\tau} \oplus H^{-,\mathrm{ext}}_{\alpha,\tau}$$

where $H^{-,\omega}_{\alpha,\tau}$, $\omega = \mathrm{int}, \mathrm{ext}$, are the operators associated with the forms $\eta^{-,\omega}_{\alpha,\tau}$ acting as (3.13) and with the domains $W^{2,1}\big(\mathrm{D}^\omega_\tau, \mathrm{d}\Omega\big)$. Since $H^-_{\alpha,\tau} \geqslant H^{-,\mathrm{int}}_{\alpha,\tau} \oplus H^{-,\mathrm{ext}}_{\alpha,\tau}$, and the spectrum of $H^{-,\mathrm{int}}_{\alpha,\tau}$ is purely discrete [Da] we obtain by the minimax principle

$$\inf \sigma_{\mathrm{ess}}(H^-_{\alpha,\tau}) \geqslant \inf \sigma_{\mathrm{ess}}\big(H^{-,\mathrm{ext}}_{\alpha,\tau}\big). \tag{4.5}$$

Thus to verify the claim it suffices to check $\inf \sigma_{\mathrm{ess}}\big(H^{-,\mathrm{ext}}_{\alpha,\tau}\big) \geqslant \epsilon(\alpha)$. By the assumption (a$\Gamma$2) the quantities $m^+_\tau := \sup_{\Gamma^{\mathrm{ext}}_\tau} \xi$ and $m^-_\tau := \inf_{\Gamma^{\mathrm{ext}}_\tau} \xi$ tend to one as $\tau \to \infty$. Using (3.13), (4.2), and the block form of $G^{ij}$ we get the estimate

$$\begin{aligned}
\eta^{-,\mathrm{ext}}_{\alpha,\tau}[\psi] &\geqslant \int_{\mathrm{D}^{\mathrm{ext}}_\tau} |\partial_3 \psi(q)|^2 \,\mathrm{d}\Omega - \alpha \int_{\Gamma^{\mathrm{ext}}_\tau} |\psi(s, 0)|^2 \,\mathrm{d}\Gamma \\
&\geqslant m^-_\tau \int_{\mathrm{D}^{\mathrm{ext}}_\tau} |\partial_3 \psi(q)|^2 \,\mathrm{d}\Gamma \,\mathrm{d}u - \alpha \int_{\Gamma^{\mathrm{ext}}_\tau} |\psi(s, 0)|^2 \,\mathrm{d}\Gamma \\
&\geqslant \varepsilon_\tau \int_{\mathrm{D}^{\mathrm{ext}}_\tau} |\psi(q)|^2 \,\mathrm{d}\Omega
\end{aligned}$$

where $\varepsilon_\tau := \frac{\alpha^2}{m^+_\tau m^-_\tau}\big[-\frac{1}{4} - C_N \exp\big(-\frac{1}{2}\frac{\alpha}{m^-_\tau} d\big)\big]$. Since $\tau$ is an arbitrary parameter we get the inclusion (4.1) with $\epsilon(\alpha) = -\frac{\alpha^2}{4} - C_N \alpha^2 \,\mathrm{e}^{-\alpha d/2}$. $\square$

**Remark 4.2.**

(a) The assumption (a$\Gamma$2) can be replaced by a hypothesis about the normal vector to $\Gamma$, namely

$$(\mathrm{a}\Gamma 2') \quad n \to n_0 \text{ as } r \to \infty, \text{ where } n_0 \text{ is a fixed vector.}$$

It is easy to see that the latter implies (a$\Gamma$2); to this end one has to combine (3.2) and (3.3). Of course, the converse statement is not true: for example, the elliptic paraboloid satisfies (a$\Gamma$2) but not (a$\Gamma$2$'$). As we will see in the further discussion, the assumption (a$\Gamma$2$'$) implies at the same time (a$\Gamma$1). Let us show that the claim of theorem 4.1 can be strengthened in this situation. Specifically, if $\alpha > 0$ and the surface $\Gamma$ satisfies (a$\Gamma$2$'$), then

$$\sigma_{\mathrm{ess}}(H_\alpha) \subseteq \left[-\frac{\alpha^2}{4}, \infty\right).$$

To prove this we have to show that

$$\inf \sigma_{\mathrm{ess}}(H_\alpha) \geqslant -\frac{\alpha^2}{4}. \tag{4.6}$$

Similarly as in the proof of theorem 4.1 we divide the surface $\Gamma$ into two components $\Gamma_\tau^{\mathrm{int}}$ and $\Gamma_\tau^{\mathrm{ext}}$.

Let us assume for a moment that $d(\tau) \equiv d_\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is an arbitrary function and consider a one-parameter family of maps $\mathcal{L}_\tau : \mathrm{D}_{d_\tau} \to \Omega_{d_\tau}$. Using (a$\Gamma$2$'$) and the mean value theorem we may show that the number ensuring the injectivity of $\mathcal{L}_\tau$ in $\mathrm{D}_{d_\tau}^{\mathrm{ext}} = \{(s, u) : s \in \Gamma_\tau^{\mathrm{ext}}, u \in (-d_\tau, d_\tau)\}$ is proportional to the expression $Q(\tau) \equiv \inf_{s_1, s_2 \in \Gamma_\tau^{\mathrm{ext}}} \frac{|s_1 - s_2|}{|n(s_1) - n(s_2)|}$ and

$$Q(\tau) \to \infty \quad \text{as} \quad \tau \to \infty.$$

This particularly means that the assumption (a$\Gamma$2$'$)—in distinction to the weaker hypothesis (a$\Gamma$2)—implies (a$\Gamma$1) as we have indicated above. Furthermore, under (a$\Gamma$2$'$) we can find $d_\tau$ such that $d_\tau \to \infty$ as $\tau \to \infty$ and the maps $\mathcal{L}_\tau$ are injective in $\mathrm{D}_{d_\tau}^{\mathrm{ext}}$. Relying on the same arguments as in the previous proof we can show that to get (4.6) it suffices to check

$$\inf \sigma_{\mathrm{ess}}\big(H_{\alpha, d_\tau}^{-,\mathrm{ext}}\big) \geqslant -\frac{\alpha^2}{4}$$

where $H_{\alpha, d_\tau}^{-,\mathrm{ext}}$ is associated with the form $\eta_{\alpha, d_\tau}^{-,\mathrm{ext}}$ acting as (3.13) and with the domain $W^{2,1}(\mathrm{D}_{d_\tau}, \mathrm{d}\Omega)$. According to the previous discussion the assumption (a$\Gamma$2$'$) implies $\sup_{\Gamma_\tau^{\mathrm{ext}}} |M|$, $\sup_{\Gamma_\tau^{\mathrm{ext}}} |K| \to 0$, so we can always choose $d_\tau$ such that $n_\tau^+ \equiv \sup_{\mathrm{D}_\tau^{\mathrm{ext}}} \xi$ and $n_\tau^- \equiv \inf_{\mathrm{D}_\tau^{\mathrm{ext}}} \xi \to 1$ as $\tau \to \infty$ (recall that we can choose $d_\tau$ to go to infinity in an arbitrarily slow way). Mimicking now the calculations from the proof of theorem 4.1 we arrive at

$$\eta_{\alpha, d_\tau}^{-,\mathrm{ext}}[\psi] \geqslant \tilde{\varepsilon}_\tau \int_{\mathrm{D}_{d_\tau}^{\mathrm{ext}}} |\psi(q)|^2 \, \mathrm{d}\Omega$$

where $\tilde{\varepsilon}_\tau := \frac{\alpha^2}{n_\tau^+ n_\tau^-}\big(-\frac{1}{4} - C_N \exp\big(-\frac{1}{2}\frac{\alpha}{n_\tau^-} d_\tau\big)\big)$. Since $\tilde{\varepsilon}_\tau \to -\frac{\alpha^2}{4}$ as $\tau \to \infty$ and $\tau$ is an arbitrary parameter we get the stated inequality (4.6).

(b) Note that the assumption (a$\Gamma$2$'$) implies the uniform ellipticity (3.9). Indeed, it means that for each $\varepsilon > 0$ there is a compact $\Sigma_\varepsilon$ such that the inequality $|n(s) - n(s_0)| < \varepsilon$ holds for all $s, s_0 \in \mathbb{R}^2 \backslash \Sigma_\varepsilon$. Without loss of generality we can fix $s_0$ and suppose that $n(s_0) = (0, 0, 1)$, then $\Gamma$ coincides outside $\Sigma_\varepsilon$ with the graph of a smooth function $f$. In that case we have explicit expressions $n = g^{-1/2}(-f_1, -f_2, 1)$ and

$$(g_{\mu\nu}) = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix}$$

where $f_j \equiv \partial_j f$, which imply $\max\{f_1, f_2\} \leqslant \varepsilon(1-\varepsilon^2)^{-1/2}$, and thus (3.9) outside $\Sigma_\varepsilon$. On the other hand, the eigenvalues of $(g_{\mu\nu})$ are continuous functions of the parameters and thus they reach their maxima and minima in $\Sigma_\varepsilon$.

*4.2. Existence of bound states and asymptotics of the eigenvalues*

In order to show the existence of bound states and to derive the asymptotic behaviour of the eigenvalues of $H_\alpha$ we employ the 'comparison' operator

$$S := -\Delta_\Gamma + K - M^2 : D(-\Delta_\Gamma) \to L^2(\mathbb{R}^2, \mathrm{d}\Gamma)$$

where $-\Delta_\Gamma$ is the Beltrami–Laplace operator given by

$$-\Delta_\Gamma = -g^{-1/2}\partial_\mu g^{1/2} g^{\mu\nu}\partial_\nu$$

and $D(-\Delta_\Gamma)$ is its maximal domain in $L^2(\mathbb{R}^2, \mathrm{d}\Gamma)$. By (3.9) and the assumption (a$\Gamma$2) it is straightforward to check that $S$ is a well-defined self-adjoint in $L^2(\mathbb{R}^2, \mathrm{d}\Gamma)$ and its domain coincides with $W^{2,2}(\mathbb{R}^2)$. Moreover, using (3.4) we infer that $K - M^2$ is an attractive potential vanishing at infinity. Applying again (3.9) and the standard results for a two-dimensional Schrödinger operator (see, for example, [Si]) we may conclude that $\sigma_{\mathrm{ess}}(S) = [0, \infty)$, and that $S$ has at least one negative eigenvalue provided $\Gamma$ is not a plane.

We denote by $\mu_j$ the $j$th eigenvalue of $S$ counted with multiplicity, $j = 1, \ldots, N$ with $1 \leqslant N \leqslant \infty$. Our main result looks as follows.

**Theorem 4.3.** *Let* $\Gamma$ *satisfy assumptions* (a$\Gamma$1)*, (3.9) and* (a$\Gamma$2)*, or alternatively* (a$\Gamma$2$'$)*. Unless* $\Gamma$ *is a plane, there exists at least one isolated eigenvalue of* $H_\alpha$ *below the threshold of the essential spectrum for* $\alpha$ *large enough. Moreover, the eigenvalues* $\lambda_j(\alpha)$ *of* $H_\alpha$ *have the following asymptotic expansion:*

$$\lambda_j(\alpha) = -\tfrac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1}\log\alpha) \qquad \text{as} \quad \alpha \to \infty.$$

As the first step to prove the theorem we derive some useful estimates. Consider again the layer neighbourhood $\Omega_d$ of $\Gamma$ and quadratic forms $\eta_\alpha^\pm$ acting as (3.13) with the domains $W_0^{2,1}(\mathrm{D}_d, \mathrm{d}\Omega)$ and $W^{2,1}(\mathrm{D}_d, \mathrm{d}\Omega)$, respectively. As we have already mentioned, due to the Dirichlet–Neumann bracketing the associated operators $H_\alpha^+$, $H_\alpha^-$ living in $L^2(\mathrm{D}_d, \mathrm{d}\Omega)$ give the upper and lower estimates for negative eigenvalues of $H_\alpha$. Moreover, in view of the unitary equivalence of $H_\alpha^\pm$ and $B_\alpha^\pm$ we can consider $B_\alpha^\pm$ instead of $H_\alpha^\pm$.

Since $B_\alpha^\pm$ are still not quite easy to handle, our next aim is to estimate them by operators with separated variables. Using the explicit form of the potential $V_1$, and taking into account (3.7), (3.8), (3.9) together with assumption (a$\Gamma$2) we can find the numbers $v^+$, $v^-$ such that

$$dv^- \leqslant V_1 \leqslant dv^+ \tag{4.7}$$

holds for $d < \varrho$. On the other hand, we can estimate the potential $V_2$ by

$$C_-^{-2}(d)(K - M^2) \leqslant V_2 \leqslant C_+^{-2}(d)(K - M^2) \tag{4.8}$$

where $C_\pm$ were introduced in (3.7). Now we define another pair of estimating operators in $L^2(\mathbb{R}^2, \mathrm{d}\Gamma) \otimes L^2(-d, d)$ by

$$\tilde{B}^+_{\alpha,d} := U_d^+ \otimes 1 + 1 \otimes T^+_{\alpha,d} \qquad \tilde{B}^-_{\alpha,d} := U_d^- \otimes 1 + 1 \otimes T^-_{\alpha,d} \tag{4.9}$$

where

$$U_d^\pm = -C_\pm(d)\Delta_\Gamma + C_\pm^{-2}(d)(K - M^2) + v \pm d$$

and $T^{\pm}_{\alpha,d}$ are associated with the quadratic forms given by

$$t^{+}_{\alpha,d}[\psi] = \int_{-d}^{d} |\partial_3\psi|^2 \,\mathrm{d}u - \alpha|\psi(0)|^2$$

$$t^{-}_{\alpha,d}[\psi] = \int_{-d}^{d} |\partial_3\psi|^2 \,\mathrm{d}u - \alpha|\psi(0)|^2 - D_d(|\psi(d)|^2 + |\psi(-d)|^2)$$

for $\psi \in W^{2,1}_0(-d,d)$ and $W^{2,1}(-d,d)$, respectively. The coefficient $D_d := 2(\|M\|_\infty + \|K\|_\infty d)$ coming from the Neumann boundary conditions is in distinction to the similar quantity for $B^{-}_{\alpha}$ independent of the surface variables $s$. It is clear from (4.7) and (4.8) that $\tilde{B}^{\pm}_{\alpha,d}$ give the sought bounds for $B^{\pm}_{\alpha}$, i.e.

$$B^{+}_{\alpha} \leqslant \tilde{B}^{+}_{\alpha,d} \qquad \text{and} \qquad \tilde{B}^{-}_{\alpha,d} \leqslant B^{-}_{\alpha}. \tag{4.10}$$

Since $\tilde{B}^{\pm}_{\alpha,d}$ have separated variables their spectra express through those of their 'constituent' operators. Consider first the 'surface' part. Repeating the arguments we have used for the operator $S$ we may conclude that $\sigma_d\left(U^{\pm}_d\right) \neq \emptyset$. Our aim is now to find the asymptotic behaviour of the eigenvalues $\mu^{\pm}_j(d)$ of $U^{\pm}_d$ for small $d$, or more precisely, we would like to show that numbers $\mu_j(d)$ approach the eigenvalues $\mu_j$ of the comparison operator $S$ as $d \to 0$.

**Lemma 4.4.** *The eigenvalues of $U^{\pm}_d$ have the following asymptotics*

$$\mu^{\pm}_j(d) = \mu_j + C^{\pm}_j d + \mathcal{O}(d^2) \tag{4.11}$$

*for $d \to 0$, where $C^{\pm}_j$ are constants.*

**Proof.** Assume $d < \varrho$. Applying the explicit form for $U^{+}_d$ we immediately get

$$U^{+}_d - C_{+}(d)S = v^{+}d + \left(C^{-2}_{+}(d) - C_{+}(d)\right)(K - M^2). \tag{4.12}$$

Using the formula for $C_{+}(d)$ it is easy to check that any function $d \mapsto m(d)$ with the asymptotic behaviour

$$\left(v^{+} + \left(\|K\|_\infty + \|M\|^2_\infty\right)\varrho^{-1}\right)d + \mathcal{O}(d^2)$$

as $d \to 0$, gives the upper bound for the rhs of (4.12) for sufficiently small $d$. Thus we get the following estimate:

$$\left\| U^{+}_d - C_{+}(d)S \right\| \leqslant m(d).$$

Combining this inequality and the minimax principle we get

$$\left|\mu^{+}_j(d) - C_{+}(d)\mu_j\right| \leqslant m(d)$$

which implies

$$\left|\mu^{+}_j(d) - \mu_j\right| \leqslant m(d) + d|(2\varrho^{-1} + \varrho^{-2}d)\mu_j|.$$

and thus (4.11) for $\mu^{+}_j(d)$. The proof for $\mu^{-}_j(d)$ is analogous. □

Let us pass to the transverse part. To estimate the negative eigenvalues of $T^{\pm}_{\alpha,d}$ we will use the lemma from [EY1].

**Lemma 4.5.** *There exist positive constants $C$, $C_N$ such that each of the operators $T^{\pm}_{\alpha,d}$ has a single negative eigenvalue $\kappa^{\pm}_{\alpha,d}$ satisfying*

$$-\frac{\alpha^2}{4} - C_N\alpha^2\,\mathrm{e}^{-\alpha d/2} < \kappa^{-}_{\alpha,d} < -\frac{\alpha^2}{4} < \kappa^{+}_{\alpha,d} < -\frac{\alpha^2}{4} + 2\alpha^2\,\mathrm{e}^{-\alpha d/2}$$

*for any $\alpha > C\max\{d^{-1}, D_d\}$.*

Now we are ready to prove theorem 4.3. Put

$$d = d(\alpha) := 6\alpha^{-1}\log\alpha. \tag{4.13}$$

From lemma 4.5 we know that operators $T^{\pm}_{a,d(\alpha)}$ have single negative eigenvalues $\kappa^{\pm}_{\alpha} \equiv \kappa^{\pm}_{\alpha,d(\alpha)}$ and in view of the decomposition (4.9) we infer that $\kappa^{\pm}_{\alpha} + \mu^{\pm}_{j,\alpha}$, where $\mu^{\pm}_{j,\alpha} = \mu^{\pm}_{j}(d(\alpha))$, are eigenvalues of $\tilde{B}^{\pm}_{\alpha}$. Using again lemma 4.5 together with lemma 4.4 we may conclude that $\kappa^{\pm}_{\alpha} + \mu^{\pm}_{j}(\alpha)$ have the following asymptotics:

$$\kappa^{\pm}_{\alpha} + \mu^{\pm}_{j,\alpha} = -\tfrac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1}\log\alpha).$$

Finally, it follows from (3.12) and (4.10) that the same asymptotics holds for eigenvalues of $H_\alpha$. This completes the proof.

**Remark 4.6.** The existence of isolated points of the spectrum below the threshold of the essential spectrum for large $\alpha$ can be alternatively proved by constructing a trial function $\psi \in D(H_\alpha)$ such that the inequality

$$(H_\alpha\psi, \psi) < \left(-\frac{\alpha^2}{4} - D\right)\|\psi\|^2$$

holds with a positive $D$. By Dirichlet bracketing it suffices to find a function $f \in D\big(H^{+}_{\alpha}\big)$ satisfying

$$\big(H^{+}_{\alpha} f, f\big)_G < \left(-\frac{\alpha^2}{4} - D\right)\|f\|^2_G. \tag{4.14}$$

Using the unitary equivalence $H^{+}_{\alpha}$ and $B^{+}_{\alpha}$ and inequality $B^{+}_{\alpha} \leqslant \tilde{B}^{+}_{\alpha,d}$ we infer that (4.14) is satisfied if

$$\big(\tilde{B}^{+}_{\alpha,d}h, h\big)_g + \frac{\alpha^2}{4}\|h\|^2_g < -D\|h\|^2_g \qquad h = \hat{U} f \tag{4.15}$$

holds. Let $h$ be a radially symmetric function given by $h(r,u) = \phi(r)\chi(u)$ where $\phi$ is arbitrary for a moment and $\chi$ is the normalized eigenfunction of $T^{+}_{\alpha,d}$ corresponding to the negative eigenvalues $\kappa^{+}_{\alpha,d}$ (see lemma 4.5). Substituting again $d$ from (4.13) one can show that there exist functions $\theta_1(\cdot), \theta_2(\cdot)$ and $\beta(\cdot)$ such that $\theta_1(\alpha), \theta_2(\alpha) \to 1$, $\beta(\alpha) \to 0$ as $\alpha \to \infty$ and the expression

$$\theta_1(\alpha)\|\nabla\phi\|'^2_g + \beta(\alpha)\|\phi\|'^2_g + \theta_2(\alpha)((K - M^2)\phi, \phi)'_g \tag{4.16}$$

gives the upper bound for the lhs of (4.15). Here we use the notation $(\cdot,\cdot)'_g$ for the scalar product in $L^2(\mathbb{R}^2, \mathrm{d}\Gamma)$. Given $r_0 > 0$ we define

$$\phi_\sigma(r) := \min\left\{1, \frac{K_0(\sigma r)}{K_0(\sigma r_0)}\right\} \qquad \sigma > 0$$

where $K_0$ is the Macdonald function, and substitute $\phi_\sigma$ for $\phi$ in the above formulae. Since $K_0$ is strictly decreasing the function $\phi_\sigma$ is not smooth at $r = r_0$, and consequently, $\phi_\sigma\chi$ does not belong to the domain of operator $\tilde{B}^{+}_{\alpha,d}$. However, it belongs to $W^{2,1}_0(\mathrm{D}_d, \mathrm{d}\Gamma\,\mathrm{d}u)$ which coincides with the domain of the form associated with $\tilde{B}^{+}_{\alpha,d}$. Therefore, it suffices to show that the quantity (4.16) (with the replacement $\phi \mapsto \phi_\sigma$) is smaller than $-D\|\phi_\sigma\|'^2_g$. Using the properties of the Macdonald function one can show that $\|\nabla\phi_\sigma\|'_g \to 0$ and $\phi_\sigma(r) \to 1$, pointwise as $\sigma \to 0$ (cf [EV]). Moreover, since the expression $K - M^2$ is strictly negative in some open subset of $\mathbb{R}^2$ there exist positive constants $\sigma_0$, $\alpha_0$ and $C = C(\sigma_0, \alpha_0)$ such that

$$\theta_1(\alpha)\big\|\nabla\phi_{\sigma_0}\big\|'^2_g + \theta_2(\alpha)\big((K - M^2)\phi_{\sigma_0}, \phi_{\sigma_0}\big)'_g < -C$$

holds for all $\alpha > \alpha_0$. Choosing $\tilde{D} < C\|\phi_{\sigma_0}\|_g'^{-2}$ and using the fact that $\beta(\alpha) \to 0$ as $\alpha \to \infty$ we can always find $\tilde{\alpha} > \alpha_0$ such that the inequality

$$\theta_1(\alpha)\|\nabla\phi_{\sigma_0}\|_g'^2 + \beta(\alpha)\|\phi_{\sigma_0}\|_g'^2 + \theta_2(\alpha)\big((K - M^2)\phi_{\sigma_0}, \phi_{\sigma_0}\big)_g' < -\tilde{D}\|\phi_{\sigma_0}\|_g'^2$$

holds for all $\alpha \geqslant \tilde{\alpha}$. Putting $D = \tilde{D}$ we get the claim. Let us stress that the above argument does not require the assumption (3.9) about the uniform ellipticity of the metric tensor $g_{\mu\nu}$.

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## Appendix. Proof of theorem 3.2

Since the argument is analogous to that of theorem 4.1 in [EI] we here present only the main steps. Given $z \in \varrho(-\Delta)$ we denote by $\mathrm{R}^0(z)$ the free resolvent, $\mathrm{R}^0(z) = (-\Delta - z)^{-1}$, which is an integral operator with the kernel

$$\mathrm{G}_z(x, y) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}p(x-y)}}{p^3 - z} = \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}\sqrt{z}|x-y|}}{|x - y|}. \tag{A.1}$$

It suffices for further discussion to consider only a subset of $\varrho(-\Delta)$, for instance, the negative halfline. Thus we put $z = k^2$, where $k = \mathrm{i}\kappa$ with $\kappa > 0$, and introduce the notation $R^0(k) = \mathrm{R}^0(k^2)$, $G_k = \mathrm{G}_{k^2}$.

First we express the resolvent of $H_d(W)$ in the Birman–Schwinger form

$$\begin{aligned} R^d(k) &:= (H_d(W) - k^2)^{-1} \\ &= R^0(k) + R^0(k)V_d^{1/2}\big[I + |V_d|^{1/2}R^0(k)V_d^{1/2}\big]^{-1}|V_d|^{1/2}R^0(k) \end{aligned} \tag{A.2}$$

for $k^2 \in \varrho(-\Delta) \cap \varrho(H_d(W))$, with the usual $V_d^{1/2} = |V_d|^{1/2}\mathrm{sgn}V_d$. Mimicking the analysis in [EI] the second term on the rhs of (A.2) can be written as the product of integral operators $B_d(I - C_d)^{-1}\tilde{B}_d$ mapping $L^2(\mathbb{R}^3) \to L^2(\mathrm{D}_1, \mathrm{d}\Gamma\,\mathrm{d}t) \to L^2(\mathrm{D}_1, \mathrm{d}\Gamma\,\mathrm{d}t) \to L^2(\mathbb{R}^3)$ with the kernels

$$\begin{aligned} B_d(x; s', t') &= G_k(x - x'(s', \mathrm{d}t'))\xi(s', \mathrm{d}t')W(t)^{1/2} \\ \tilde{B}_d(s, t; x') &= |W(t)|^{1/2}\xi(s, \mathrm{d}t)G_k(x' - x(s, \mathrm{d}t)) \\ C_d(s, t; s', t') &= |W(t)|^{1/2}G_k(x(s, \mathrm{d}t) - x'(s', \mathrm{d}t'))W(t')^{1/2} \end{aligned}$$

where $x(s, \mathrm{d}t) \equiv \gamma(s) + \mathrm{d}t\, n(s)$. Using (A.1) we estimate $\|C_d\| \leqslant \|W\|_\infty|k|^{-2}$; thus $\|C_d\| < 1$ holds, uniformly in $d$, for all sufficiently large $\kappa$. Thus we can expand the operator under consideration as a geometric series,

$$B_d(I - C_d)^{-1}\tilde{B}_d = \sum_{j=0}^{\infty} B_d C_d^j \tilde{B}_d. \tag{A.3}$$

Let us turn to the analysis of the resolvent of $H_\alpha$. With this aim we need the embedding operators associated with $R^0(k)$, namely

$$R^0_\mu(k) := I_\mu R^0(k) : L^2(\mathbb{R}^3) \to L^2(\mu)$$

with the adjoint $\big[R^0_\mu(k)\big]^* : L^2(\mu) \to L^2(\mathbb{R}^3)$, and

$$R^0_{\mu\mu}(k) := I_\mu\big[R^0_\mu(k)\big]^* : L^2(\mu) \to L^2(\mu).$$

Using this notation one can express the resolvent of $H_\alpha$ as

$$\begin{aligned} R(k) &:= (H_a - k^2)^{-1} \\ &= R^0(k) + \alpha\big[R^0_\mu(k)\big]^*\big[I - \alpha R^0_{\mu\mu}(k)\big]^{-1} R^0_\mu(k) \end{aligned} \tag{A.4}$$

for any $k^2 \in \varrho(-\Delta) \cap \varrho(H_\alpha)$ (cf [BEKŠ]). Using $\alpha = \int_{-1}^{-1} W(t)\,\mathrm{d}t$ we can expand the second term on the rhs of (A.4) as

$$\sum_{j=0}^{\infty} BC^j \tilde{B} \tag{A.5}$$

where $BC^j\tilde{B}$ is the product of operators mapping $L^2(\mathbb{R}^3) \to L^2(D_1, \mathrm{d}\Gamma\,\mathrm{d}t) \to L^2(D_1, \mathrm{d}\Gamma\,\mathrm{d}t) \to L^2(\mathbb{R}^3)$ with the kernels

$$\begin{aligned} &B(x; s', t') = G_k(x - \gamma(s'))W(t)^{1/2} \\ &\tilde{B}(s, t; x') = |W(t)|^{1/2} G_k(x' - \gamma(s)) \\ &C(s, t; s', t') = |W(t)|^{1/2} G_k(\gamma(s) - \gamma(s'))W(t')^{1/2}. \end{aligned}$$

Applying (A.3) and (A.5) and repeating the argument of [EI] we can estimate

$$\|R^d(k) - R(k)\| \leqslant \sum_{j=0}^{\infty} \big\|BC^j\tilde{B} - B_d C_d^j \tilde{B}_d\big\| \leqslant c(\|B_d - B\| + \|\tilde{B}_d - \tilde{B}\| + \|C_d - C\|) \tag{A.6}$$

for a positive $c$. The first norm on the rhs of (A.6) can be estimated by

$$\|W\|_\infty^{1/2}\{\|\breve{R}_{d,1}(k) - \breve{R}_0(k)\| + d(2\|M\|_\infty + d\|K\|_\infty)\|\breve{R}_{d,1}(k)\|\}$$

where $\breve{R}_{d,1}(k)$, $\breve{R}_0(k)$ are the integral operators acting from $L^2(\mathrm{D}_1, \mathrm{d}\Gamma\,\mathrm{d}t)$ to $L^2(\mathbb{R}^3)$ with the kernels $G_k(x - x'(s', t'))$ and $G_k(x - \gamma(s'))$. Since $\|\breve{R}_{d,1}(k)\|$ is uniformly bounded w.r.t. $d$, the norm convergence in question for $d \to 0$ will follow from the corresponding property of $\|\breve{R}_{d,1}(k) - \breve{R}_0(k)\|$. The latter can be estimated by the Schur–Holmgren bound,

$$\|\breve{R}_{d,1}(k) - \breve{R}_0(k)\| \leqslant (h_1 h_\infty)^{1/2}$$

where

$$\begin{aligned} h_\infty &:= \sup_{x\in\mathbb{R}^3} \int_{\mathrm{D}_1} |(\breve{R}_{d,1}(k) - \breve{R}_0(k))(x, x'(s', \mathrm{d}t'))|\,\mathrm{d}\Gamma'\,\mathrm{d}t' \\ h_1 &:= \sup_{x'\in\mathrm{D}_1} \int_{\mathrm{D}_1} |(\breve{R}_{d,1}(k) - \breve{R}_0(k))(x, x')|\,\mathrm{d}x. \end{aligned}$$

Finally, using the mean value theorem in combination with the fact that $|G_k'| \in L^1(\mathbb{R}^3, g^{1/2}\,\mathrm{d}x)$, where $|G_k'(x)| = \frac{1}{4\pi}\frac{(\kappa|x|+1)}{x^2}\,\mathrm{e}^{-\kappa|x|}$, one can establish the existence of a constant $c_1$ such that $h_1, h_\infty$ entering the SH-bound are both majorized by the expression $c_1 d\|G_k'\|_{L^2(\mathbb{R}^3, g^{1/2}\,\mathrm{d}x)}$. The convergence of $\|\tilde{B}_d - \tilde{B}\|$ and $\|C_d - C\|$ is checked in the same way.

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